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A MIXING LAYER IN A HOMOGENEOUS FLUID

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A mathematical model for the evolution of a mixing layer in shear flows is constructed. The problem of a mixing layer with pressure gradient is solved; in particular, the distributions of the velocity and basic characteristics of turbulent flow in the mixing layer are obtained.

1. TURBULENCE MODELS

The Prandtl [1] and Taylor [2] models, which are based on the "gradient hypothesis," and their development with the equations for the turbulent energy and the scale of turbulence [3] are widely applied to calculation of shear flows of a homogeneous fluid. The hypothesis of the proportionality of the Reynolds stresses to the kinetic turbulence energy in a developed turbulent layer underlies the Townsend [4] and Bradshaw [5] "hyperbolic" models. It is noteworthy that even in the case of an incompressible fluid, the flows with free (internal and external) boundaries have a number of properties characteristic of the solutions of hyperbolic systems of equations. The boundaries that divide the regions of potential and turbulent flows are very distinct, and the flow perturbations related to deformation of these boundaries propagate in a wavy manner with a finite velocity. The waves at the interfaces are responsible for one more specific feature of flows, namely, flow intermittency [6]. In the semiempirical theories of turbulence, this effect is usually related to turbulent diffusion and is simulated by appropriate "diffusion" terms in the energy equation. The resulting equations represent a complex nonlinear system for which even the construction of self-similar solutions by analytical methods is difficult. For a homogeneous fluid, there is a limited number of exact solutions, mainly, for the simplest models, which are based on the hypothesis of a mixing length. Among them are Tollmien and Görtler's solutions for mixing layers and jets and Schlichting and Taylor's solutions for wakes [7].

The goal of this study is to analyze quite simple equations that describe a nonstationary interaction between the mean flow and small-scale fluid motions. The basic system considered is obtained by means of further simplification of well-known models [4–6, 8].

The problem of the evolution of tangent discontinuity in a homogeneous fluid is considered in [9]. It has been shown that the nonlinear hyperbolic system constructed describes the process of transverse momentum transfer by "large vortices" generated by the velocity shear. The solution of this system also determines the spread rate of a turbulized fluid in undisturbed flow. The average flow characteristics with allowance for intermittency are found from the solution of a linear hyperbolic system as soon as the distributions of the turbulent velocity and energy components are found. Despite an increased number of equations, their structure is fairly simple to find explicitly a self-similar solution describing the decay of tangent discontinuity in a homogeneous fluid. In [10], this analysis is applied to steady-state flows, and a self-similar solution for a mixing layer is constructed.

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Ovsyannikov et al. [11. Chapter 4] used a similar approach for stratified flows. The turbulent-intermixing model constructed in [11] describes the interaction between large vortices in homogeneous layers and interface waves and the mean flow. A linear analysis of the evolution of perturbations of this system in steady-state "slipping" flow shows that the amplitude of small-scale oscillations increases in any shear flow. This gives rise to "slippage" of homogeneous layers relative to the interlayers and creates conditions for excitation of interface waves. Wave generation in the thermoline can also be described within the framework of this model.

Here we construct a combined turbulent-intermixing model that is suitable for treating the evolution of a mixing layer with pressure gradient. One basic distinguishing feature of this model is that the mean-flow parameters in the intermediate layer are first determined by solving a nonlinear system of equations and the distributions of the average-velocity and turbulent-flow characteristics across the mixing layer are then restored from the solution of a semilinear system of equations.

2. FREE SHEAR TURBULENCE IN A HOMOGENEOUS FLUID

Boundary-Layer Approximation. The plane-parallel motions of an effectively inviscid homogeneous incompressible fluid ($\rho \equiv 1$) are considered. Let the vertical average-velocity component v be small compared with the horizontal component u. Then, in the presence of the vertical velocity shear of u in flow, the momentum is transferred along the vertical by the Reynolds stress $\tau = -\overline{u'v'}$. In the developed turbulent flow, the dependence [6]

$$\tau = \sigma q^2 \qquad (\sigma = \sigma_0 \operatorname{sign} u_y, \quad \text{where} \quad \sigma_0 = \text{const}), \tag{2.1}$$

corresponds to experimental data. Here $q^2 = \overline{u'^2 + v'^2 + w'^2}$ is the kinetic energy of the pulsatory motion (u', v', and w' are the horizontal, vertical, and transverse pulsatory velocity components). Because the quantity σ_0 is small ($\sigma_0 \approx 0.15$), the transfer velocity in the vertical direction is also small compared to the velocity in the horizontal direction and one can pass to the boundary-layer approximation by means of the following expansion of the variables:

$$x \to x, \quad y \to \sigma_0 y, \quad t \to \sigma_0^{-1/2} t, \quad u \to \sigma_0^{1/2} u, \quad v \to \sigma_0^{3/2} v, \quad p \to \sigma_0 p, \quad q \to \sigma_0^{1/2} q.$$

Here p is the pressure, t is the time, and x and y are the horizontal and vertical coordinates. The "diffusion terms" in the energy equation are regarded as higher-order quantities relative to σ_0 in comparison with q^3:

$$\overline{(u'^2 + v'^2 + w'^2)u' + pu'} = o(\sigma_0)q^3, \quad \overline{(u'^2 + v'^2 + w'^2)v' + pv'} = o(\sigma_0)q^3.$$

In addition, we assume that $\overline{u'^2} = \overline{v'^2} + O(\sigma_0)q^2$.

Equating terms with the same degrees σ_0 in the Reynolds equations, we obtain the boundary-layer equations

$$\begin{gathered} u_x + v_y = 0, \quad P_y = 0, \quad u_t + (u^2 + P)_x + (uv - \sigma q^2)_y = 0, \\ \Big(\frac{u^2}{2} + \frac{q^2}{2}\Big)_t + \Big(\Big(\frac{u^2}{2} + \frac{q^2}{2}\Big)u + Pu\Big)_x + \Big(\Big(\frac{u^2}{2} + \frac{q^2}{2}\Big)v + Pv - \sigma q^2 u\Big)_y = -\varepsilon. \end{gathered} \tag{2.2}$$

Here $P = \overline{p + u'^2}$ and $\sigma = \operatorname{sign} u_y$.

Because vortices of different scale are involved in turbulent motion, by the quantity q^2 we mean the energy of oriented planar vortices. Then, the quantity ε in the energy equation determines the rate of energy outflow to smaller-scale motions and can be given in the form $\varepsilon = \omega q^2 = q^3/l$, where ω is the characteristic frequency (ω^{-1} is the relaxation time) and l is the scale of turbulence. To close system (2.2), it is necessary to set the distribution of ω or l in the flow. Generally, these quantities are described by equations similar to the energy equation in (2.2). However, for the free turbulence generated by tangent discontinuity in a fluid (in particular, for the class of self-similar solutions considered below), the frequency ω in a fluid particle decreases as t^{-1}, i.e., $\omega = æt^{-1}$ ($æ = \text{const}$ and t is the time from the moment of formation of a tangent discontinuity). Thus, system (2.2) is closed by the choice of the constant æ, which characterizes the energy-transfer velocity over the spectrum, depending on the relative position of the set of vortices considered in all the eddy motions excited.

"Impurity" Transfer. Any scalar quantity φ (temperature, "impurity" concentration, density, etc.) that is preserved in laminar flow in a particle is transferred by vortices in a turbulent flow. The equations for φ and ψ can be derived, similarly to system (2.2), from the conservation equations of the quantities φ and φ^2 in a particle if the following hypothesis is adopted [4] ($\psi^2 = \overline{\varphi'^2}$):

$$-\overline{\varphi' v'} = \sigma\psi q. \tag{2.3}$$

The smallness of σ_0 allows us to apply the above-mentioned expansion to the averaged equations for $\overline{\varphi}$ and $\overline{\varphi^2}$ and obtain, under the assumption that $\overline{\varphi'^2 u'} = o(\sigma_0)\psi^2 q$, $\overline{\varphi'^2 v'} = o(\sigma_0)\psi^2 q$, and $\overline{\varphi' u'} = O(\sigma_0)\psi q$, similarly to (2.2), the equations of "impurity" transportation (the bar over φ is omitted):

$$\begin{gathered}\varphi_t + (\varphi u)_x + (\varphi v - \sigma\psi q)_y = 0, \\ \Big(\frac{\varphi^2}{2} + \frac{\psi^2}{2}\Big)_t + \Big(\Big(\frac{\varphi^2}{2} + \frac{\psi^2}{2}\Big)u\Big)_x + \Big(\Big(\frac{\varphi^2}{2} + \frac{\psi^2}{2}\Big)v - \sigma\varphi\psi q\Big)_y = -\chi.\end{gathered} \tag{2.4}$$

The dissipation rate χ of the root-mean-square fluctuations of the field is assumed to be equal to $\chi = \omega_c\varphi^2$ ($\omega_c \sim q/l$).

Thus, the "impurity" distribution in the flow can be found after construction of a solution of (2.2). The value of the expansion parameter in the model considered is fixed and not too small; therefore, discarding of these terms on the basis of the above assumptions is not quite justified.

However, the structure of the resulting equations is such that the process of turbulent diffusion is related to the phenomenon of flow intermittency in shear flow. Another specific feature of the equations obtained is that the frequency ω or ω_c is included only in the right side and characterizes the state of turbulence in the moving volume chosen, which allows us to consider the interaction between turbulent flows with different properties without additional information on the turbulence-scale distribution in the entire flow.

Flow Intermittency. As a rule, free turbulent flows are bounded by the potential-flow region. Under the action of large vortices, the distinct boundary executes oscillations with an amplitude comparable with the transverse dimension of the turbulent flow [6]. This results in intermittency with the potential motion at a fixed point of intense pulsatory motion, i.e., in flow intermittency. The intermittency coefficient λ, which characterizes the relative residence time in a completely turbulized fluid, can be used to determine the averaged stress

$$\tilde{\tau} = \lambda\sigma q^2. \tag{2.5}$$

Here q^2 is the energy of large vortices, which can be found by a conditional averaging over the region occupied by these vortices moving with the mean velocity u. Because of flow intermittency, the fixed volume-averaged velocity $\tilde{u}$ and energy $\tilde{q}^2$ differ from the turbulent components u and q^2.

Assuming that the scale of energy-containing vortices is much smaller than the scale of the vortices responsible for the vertical transfer in shear flows, i.e., the energy-containing vortices play a passive role of the "impurity," the equations for $\tilde{u}$, $\tilde{q}$, $\tilde{v}$, and $\tilde{P}$ can be introduced similarly to the "impurity"-distribution equations (2.4):

$$\tilde{\tau} = \sigma\tilde{q}q. \tag{2.6}$$

In the boundary-layer approximation, the equations for mean quantities take the form

$$\begin{gathered}\tilde{u}_x + \tilde{v}_y = 0, \qquad \tilde{P}_y = 0, \quad \tilde{u}_t + (\tilde{u}^2 + \tilde{P})_x + (\tilde{u}\tilde{v} - \sigma q\tilde{q})_y = 0, \\ \Big(\frac{\tilde{u}^2}{2} + \frac{\tilde{q}^2}{2}\Big)_t + \Big(\Big(\frac{\tilde{u}^2}{2} + \frac{\tilde{q}^2}{2}\Big)\tilde{u} + \tilde{P}\tilde{u}\Big)_x + \Big(\Big(\frac{\tilde{u}^2}{2} + \frac{\tilde{q}^2}{2}\Big)\tilde{v} + \tilde{P}\tilde{v} - \sigma q\tilde{q}\tilde{u}\Big)_y = -\tilde{\omega}\tilde{q}^2.\end{gathered} \tag{2.7}$$

System (2.7) can be solved after q is found from (2.2). The intermittency coefficient λ is determined from system (2.5), (2.6).

Horizontally Homogeneous Motions. If the mean quantities in (2.2) and (2.7) do not depend on x, in particular $P \equiv \text{const}$ and $v \equiv 0$, we obtain the system examined in [9]. It is shown in [9] that in this case, Eqs. (2.2) and (2.7) form a nonlinear hyperbolic system. The simplest problem for this class of flows is the problem of tangent-discontinuity decay. The problem is formulated as follows. Let at $t = 0$ two layers of a potential fluid move with velocities $u^{\pm}$ and their boundary be the line $y = 0$:

$$u(0, y) = \begin{cases} u^+, & y > 0, \\ u^-, & y < 0, \end{cases} \qquad q(0, y) = 0. \tag{2.8}$$

It is required to solve system (2.2), (2.8) for $t > 0$. It is noteworthy that the functions u and q defined by (2.8) are a steady solution of (2.2); however, this solution is unstable. A self-similar solution is sought in the form $u = u(\xi)$, $q = q(\xi)$, and $\omega = \alpha t^{-1}$ ($\xi = y/t$). In describing the evolution of large vortices, one can ignore their dissipation, i.e., one can set $\alpha = 0$. Here the solution of (2.2), (2.8) has a simple form:

$$u = \frac{u^+ + u^-}{2}, \qquad q = \frac{|u^+ - u^-|}{2}, \qquad |\xi| < q. \tag{2.9}$$

For $|\xi| > q$, the flow is undisturbed and $l = 2qt$ ($\sigma_0 = 1$ after expansion of the variables). The self-similar solutions $\tilde{\omega} = 2\beta q/l = \beta/t$, $\tilde{u} = \tilde{u}(\xi)$, and $\tilde{q} = \tilde{q}(\xi)$ of Eq. (2.7) with initial conditions (2.8) are also set explicitly in the region $|\xi| < q$:

$$\tilde{u}(\xi) = u + \beta q_0 \int\limits_0^{\zeta} (1 - s^2)^{\beta/2-1}\, ds, \quad \tilde{q} = q_0(1 - \zeta^2)^{\beta/2}, \quad |\zeta| < 1, \quad \zeta = \xi/q. \tag{2.10}$$

The constant q_0 is found from the condition $\tilde{u}(q) = u^+$ ensuring the continuity of the function $\tilde{u}$ on ξ. The choice of the parameter β determines the distribution of the average velocity $\tilde{u}$ and the Reynolds stress $\tilde{\tau}$. Below, the correspondence between the steady-state solutions of system (2.2), (2.7) and the nonsteady-state x-homogeneous solutions of this system is established, which permits us to choose this parameter with the use of experimental mixing-layer data.

3. PRESSURE-GRADIENT-FREE STATIONARY FLOWS

Equations of Motion. The steady-state solutions of system (2.2) without the pressure gradient ($P \equiv \text{const}$) are described by the following system:

$$u_x + v_y = 0, \quad uu_x + vu_y - (\sigma q^2)_y = 0, \quad q(uq_x + vq_y - \sigma q u_y) = -\omega q^2, \tag{3.1}$$

Here $\sigma = \operatorname{sign} u_y$. Passing to the variables x and ψ with the stream function $\partial\psi/\partial x = -v$, $\partial\psi/\partial y = u$ as an independent variable, we obtain the system

$$u(u_x - (\sigma q^2)_\psi) = 0, \qquad uq(q_x - \sigma q u_\psi) = -\omega q^2, \tag{3.2}$$

which coincides, in its basic part, with the equations of nonstationary x-homogeneous motions. The system for the averaged $\tilde{u}$ and $\tilde{q}$ is written similarly in the variables x and $\tilde{\psi}$, where $\partial\tilde{\psi}/\partial x = -\tilde{v}$ and $\partial\tilde{\psi}/\partial y = \tilde{u}$. However, for construction of a solution, it is also necessary that the conditions of matching with the undisturbed solution at the border of the potential and turbulent motions formulated in the initial variables be satisfied. The dependence of the boundary conditions on the velocity component v, eliminated from (3.2), greatly complicates the structure of the steady-state solution. We now consider the mixing-layer problem in more detail.

Mixing Layer. A steady-state, plane-parallel mixing layer forms when two homogeneous horizontal flows with velocities u^- and u^+ ($u^+ > u^-$) merge at the point $x = 0$ (Fig. 1). In contrast to the contact-discontinuity decay problem, the external flow exerts a great effect on the mixing layer. If the mixing layer

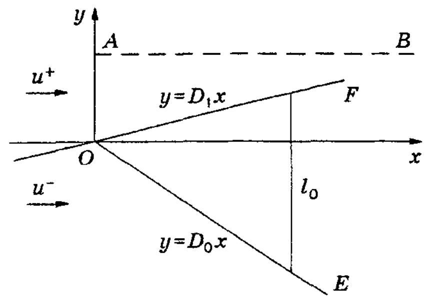

Fig. 1

FOE occurs as the initial part of a two-dimensional jet, we have $v \equiv 0$ on the axis of symmetry AB and, hence, throughout the undisturbed flow region FOAB; this leads to the following formulation of the problem ($\sigma = 1$): to find a self-similar solution $u = u(\xi)$, $v = v(\xi)$, $q = q(\xi)$, $\tilde{u} = \tilde{u}(\xi)$, $\tilde{q} = \tilde{q}(\xi)$, $\tilde{v} = \tilde{v}(\xi)$, $\xi = y/x$, and $D_0 \leqslant \xi \leqslant D_1$ of systems (2.2), (2.7) that is subject to the discontinuity conditions at the boundary $\xi = D$, $D = D_i$ ($i = 0$ and 1)

$$D[u] = [v], \qquad D[u^2] = [uv - q^2], \qquad D\Big[\Big(\frac{u^2}{2} + \frac{q^2}{2}\Big)u\Big] = \Big[\Big(\frac{u^2}{2} + \frac{q^2}{2}\Big)v - q^2u\Big],$$

$$D[\tilde{u}] = [\tilde{v}], \qquad D[\tilde{u}^2] = [\tilde{u}\tilde{v} - q\tilde{q}], \qquad D\Big[\Big(\frac{\tilde{u}^2}{2} + \frac{\tilde{q}^2}{2}\Big)\tilde{u}\Big] = \Big[\Big(\frac{\tilde{u}^2}{2} + \frac{\tilde{q}^2}{2}\Big)\tilde{v} - q\tilde{q}\tilde{u}\Big]. \qquad (3.3)$$

For $\xi > D_1$, we have $u = u^+$, $q = 0$, $v = 0$, $\tilde{u} = u^+$, $\tilde{q} = 0$, and $\tilde{v} = 0$. For $\xi < D_0$, we have $u = u^-$, $q = 0$, $\tilde{u} = u^-$, and $\tilde{q} = 0$. Here $[f] = f(D+0) - f(D-0)$, where $D = D_i$ ($i = 0$ and 1).

The boundaries of the mixing layer D_i are determined together with the solution of the problem. To construct the solution of the posed problem, the distribution of the frequencies ω and $\tilde{\omega}$ in the flow should be set. As for the tangent-discontinuity decay problem, we use the vortices comparable in scale with the mixing-layer thickness as a system of vortices executing the vertical transfer. In this case, their dissipation can be ignored, i.e., $\omega = 0$. We assume that the quantity $\tilde{\omega}$ is represented in the form $\tilde{\omega} = 2\beta q/l$ in free turbulent flows. For $\omega = 0$, system (3.2) formally coincides with the equations of nonstationary x-homogeneous intermixing; therefore, one can use the solution (2.9) of the tangent-discontinuity decay problem, i.e., to set $u = (u^- + u^+)/2$ and $q = (u^+ - u^-)/2$ for $D_0 \leqslant \xi \leqslant D_1$. The boundary conditions for these systems are in correspondence as well. As the quantity $D^\pm = \pm q$ is the discontinuity velocity for a nonstationary system, passing to the steady-state case, we have

$$D^+ = D_1u - v = D_1u^+ - v_1 = D_1u^+ = (u^+ - u^-)/2, \quad D^- = D0u^- - v_0 = D_0u - v = -(u^+ - u^-)/2$$

or

$$D_1 = \frac{u^+ - u^-}{2u^+}, \quad D_0 = -\frac{u^+ - u^-}{u^+ + u^-}\Big(\frac{u^+ - u^-}{2u^+} + 1\Big), \quad v = -\frac{(u^+ - u^-)^2}{4u^+}.$$

This solution shows that the flow boundaries are nonsymmetric relative to the x axis. The mixing layer deviates toward a slower flow. The maximum deviation is observed when the homogeneous flow outflows with velocity u^+ to the quiescent fluid. Here $D_1 = 1/2$ and $D_0 = -3/2$, or, returning to the initial variables, $D_1 = (1/2)\sigma_0$ and $D_0 = -(3/2)\sigma_0$ ($\sigma_0 = 0.15$).

Because of the nonhomogeneity of the equations for $\tilde{u}$, $\tilde{v}$, and $\tilde{q}$, the solution (2.7) subject to conditions (3.3) depends on the parameter $r = u^-/u^+$ ($0 \leqslant r \leqslant 1$). On the interval $D_0 < \xi < D_1$, this solution is determined by the system

$$\frac{dz}{d\xi} = \tilde{u}, \quad \frac{d\tilde{u}}{d\xi} = \frac{\beta uq\tilde{q}}{q^2 - z^2}, \quad \frac{d\tilde{q}}{d\xi} = -\frac{\beta uz\tilde{q}}{q^2 - z^2}. \qquad (3.4)$$

Here $z = \xi\tilde{u} - \tilde{v}$, $q = (u^+ - u^-)/2$, and $u = (u^+ + u^-)/2$. The boundary conditions for system (3.4) are relations (3.3):

$$[z] = 0, \quad [\tilde{u}] = (-1)^i[\tilde{q}] \quad \text{for} \quad \xi = D_i, \quad i = 0, 1. \tag{3.5}$$

Since the value of v and, consequently, the value of z should be given only from one side of the mixing layer by virtue of the formulation of the problem, for example for $\xi > D_1$, for system (3.4) we have three boundary conditions. The case $v = 0$ for $\xi > D_1$ is a singular case, because here we have $z(D_1 + 0) = D_1 u^+ = q$, and solution (3.4) reaches a singular point as $\xi \to D_1 - 0$ by virtue of (3.5). At the singular point, $\tilde{q} = 0$, and consequently, $[\tilde{u}] = 0$, i.e., the solution on the right side is continuous. The necessary one-constant arbitrariness is ensured by the fact that the one-parameter family of solutions reaches the singular point.

The parameter β is determined from experimental data. The value of $\beta = 6$ describes the velocity and Reynolds-stress distributions in the mixing layer with sufficient accuracy, as is shown in [10]. At the same time, the solution depends weakly on β, and, for $\beta = 4$, the self-similar solution (2.10) also shows satisfactorily the distribution of the desired quantities in the mixing layer. In particular, we have $\tilde{\tau}_{\max} = 1.18 \cdot 10^{-2}(u^+ - u^-)^2$ for $\beta = 6$ and $\tilde{\tau}_{\max} = 1.41 \cdot 10^{-2}(u^+ - u^-)^2$ for $\beta = 4$. We note that, for even natural values of the parameter β, expressions (2.10) give an explicit representation of the solution in the form of the polynomials of a self-similar variable [10].

For various values of the parameter r, the mixing-layer spread rate also can be found from solutions (3.3) and (3.4). In the model considered, it is natural to use the quantity $l_0 = (D_1 - D_0)x$ as a layer thickness, i.e., in the initial variables $dl_0/dx = 2\sigma_0 R = 0.3R$, where $R = (u^+ - u^-)/(u^+ + u^-)$. However, for comparison with experimental data, the quantity $l_1 = y_{0.95} - y_{0.1}$, where $y_{0.95}$ and $y_{0.1}$ are the values of y at which $\tilde{u} - u^- = 0.95(u^+ - u^-)$ and $\tilde{u} - u^- = 0.1(u^+ - u^-)$, respectively, is usually used.

We note that the effective mixing-layer width l_1 is a factor of 2 smaller than the maximum thickness l_0 [10].

Boundary Conditions for System (2.2). By virtue of the nonlinearity of the equations of motion, the continuous contiguity of the turbulent ($q \neq 0$) and potential ($q = 0$) solutions of system (2.2) is impossible [11]. Therefore, it is necessary to consider the discontinuous solutions. It is simple to obtain the discontinuity conditions for the general case. For our purposes, it is sufficient to obtain them for horizontally homogeneous and steady-state flows.

In the case of horizontally homogeneous flows, for $u = u(t, y)$, $q = q(t, y)$, $v \equiv 0$, and $P \equiv \text{const}$ system (2.2) has the following form ($\omega \equiv 0$):

$$u_t - (\sigma q^2)_y = 0, \quad \Big(\frac{u^2}{2} + \frac{q^2}{2}\Big)_t - \Big(\sigma u q^2\Big)_y = 0 \quad (\sigma = \operatorname{sign} u_y). \tag{3.6}$$

Equations (3.6) are a nonlinear hyperbolic system. Its characteristics are specified by the equations $dy/dt = \pm\sqrt{2}q$. The nonlinearity and hyperbolicity of Eqs. (3.6) result in the occurrence of discontinuities in the solutions, i.e., abrupt fronts dividing the regions of turbulent and potential flows. On the lines of the discontinuities propagating with velocity $D = dy/dt$, the Hugoniot conditions

$$D[u] = -[\sigma q^2], \qquad D[u^2/2 + q^2/2] = -[\sigma u q^2] \tag{3.7}$$

are satisfied. Here $\sigma = \operatorname{sign}[u]$ and $[f] = f(t, y + 0) - f(t, y - 0)$.

Functions (2.9) satisfy both system (3.6) and the discontinuity conditions (3.7). Thus, (2.9) is a generalized solution of problem (2.8). It is noteworthy that the Lax stability condition for discontinuities is also fulfilled [12]. Thus, (2.9) is a generalized stable solution of problem (2.8), (3.7) and describes the across-flow propagation of "large vortices" generated by the Kelvin–Helmholtz instability.

For steady-state flows, system (3.1) is no longer hyperbolic relative to the variable x. However, after elimination of the function v and after passage to the variables x and ψ, it becomes hyperbolic relative to this variable. Moreover, for $\omega = 0$, system (3.2) coincides with (3.6). The discontinuity conditions for system (3.1) also can be reduced to the nonstationary case. The replacement of the variable $z = Du - v$ reduces system (3.3) to the form

$$z = Du(x, y - 0) - v(x, y - 0) = Du(x, y + 0) - v(x, y + 0), \tag{3.8}$$

$$z[u] = -[\sigma q^2], \quad z[u^2/2 + q^2/2] = -[\sigma u q^2].$$

Conditions (3.8) coincide with (3.7), and the solution of the mixing-layer problem is found from solution (2.9).

4. MIXING LAYER WITH PRESSURE GRADIENT

In Secs. 2 and 3, a self-similar solution of the contact-discontinuity and pressure-gradient-free mixing-layer problems has been constructed. Here the generation of "large vortices" was determined by solution (2.9). We note that the average values of the velocities and the level of turbulence upon decaying of the contact discontinuity in a pressure-gradient-free mixing layer coincide with the exact solution (2.9). In addition, it follows from (2.9) that the rate of entrainment into the turbulent intermediate layer is proportional to the velocity of "large vortices":

$$\eta_t + (\eta \bar{u})_x = 2\sigma \bar{q}. \tag{4.1}$$

Here η is the thickness of the interlayer and $\bar{u}$ and $\bar{q}$ are the average values of the flow velocity and the velocity of "large vortices." Thus, for mixing layers, the entrainment rate (4.1) allows us to describe the properties of the solutions of the more developed model (2.2). For flows of a homogeneous or stratified fluid with pressure gradient, the vertical distribution of the velocity and energy of "large vortices" already becomes inhomogeneous within the framework of model (2.2). Nevertheless, for various types of flow, the application of the rate of fluid entrainment from homogeneous layers into a turbulent intermediate layer, which is specified by Eq. (4.1), has shown the efficiency of this approach in determining the average flow characteristics. On the basis of (4.1), a shallow-water, three-layer model that describes the evolution of a turbulent layer in the flows of homogeneous and stratified fluids with velocity shear was constructed in [13, 14].

It should be noted that the role of the shallow-water multilayer equations is not limited by the possibility of determining the average quantities in the flow. As in the case of a pressure-gradient-free mixing layer, the average velocity of "large vortices" $\bar{q}$ determines the vertical distribution of the Reynolds stresses $\tilde{\tau}$ in the flow from formula (2.6), and the turbulent-layer thickness η can be selected as a scale of turbulence l. Here the vertical-velocity and turbulent-energy distributions are described by system (2.7) in the boundary-layer approximation.

Three-Layer Model. We consider the problem of the formation of a mixing layer in a homogeneous fluid in a channel of finite depth. Let the flow be bounded by two horizontal planes, the distance between which is equal to H, and the channel be filled with a fluid. The gravity can be excluded by introducing the modified pressure $p^* = p + \rho g(H - y) = p^*(t, x)$. To describe the evolution of the average quantities in the mixing layer, we apply the three-layer, shallow-water equations, in which together with the usual shallow-water equations for homogeneous layers, the laws of conservation of total momentum and energy necessary to find the flow parameters in a turbulent interlayer [13, 14] are used.

The steady-state flow equations for the average values in the layers have the form ($\rho \equiv 1$)

$$(h^+ + \eta + h^-)_x = 0, \qquad (h^\pm u^\pm)_x = -\sigma\bar{q}, \qquad (\eta\bar{u})_x = 2\sigma\bar{q},$$

$$((u^\pm)^2/2 + p^*)_x = 0, \qquad (h^+(u^+)^2 + \eta\bar{u}^2 + h^-(u^-)^2 + p^* H)_x = 0, \tag{4.2}$$

$$(h^+(u^+)^3 + \eta\bar{u}(\bar{u}^2 + \bar{q}^2) + h^-(u^-)^3 + 2\bar{Q}p^*)_x = 0.$$

Here h^+, h^-, and η are the depths, u^+ and u^- are the velocities in the upper and lower layers, respectively, and $\bar{u}$ is the velocity in the interlayer, respectively, and $\bar{Q} = h^+u^+ + \eta\bar{u} + h^-u^- \equiv \text{const}$.

Let, for $x = 0$, the steady-state mixing layer form from two uniform flows of depth $h_0^\pm$ moving with velocity $u_0^\pm$ (Fig. 2). As a consequence of (4.2) we have the following relations:

$$h^+ + \eta + h^- = H, \quad h^+u^+ + \eta\bar{u}/2 = h_0^+u_0^+, \quad h^-u^- + \eta\bar{u}/2 = h_0^-u_0^-,$$

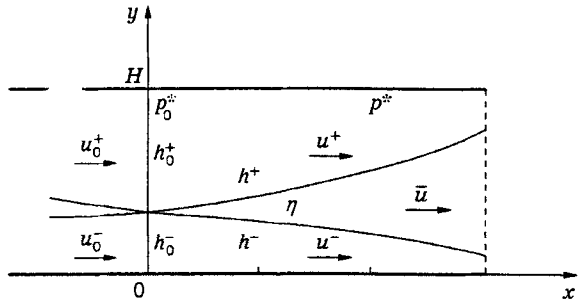

Fig. 2

$$
\begin{gathered}
(u^+)^2/2 + p^* = (u_0^+)^2/2 + p_0^*, \quad (u^-)^2/2 + p^* = (u_0^-)^2/2 + p_0^*, \\
h^+(u^+)^2 + \eta\bar{u}^2 + h^-(u^-)^2 + p^*H = h_0^+(u_0^+)^2 + h_0^-(u_0^-)^2 + p_0^*H, \\
h^+(u^+)^3 + \eta\bar{u}(\bar{u}^2 + \bar{q}^2) + h^-(u^-)^3 + 2\bar{Q}p^* = h_0^+(u_0^+)^3 + h_0^-(u_0^-)^3 + 2\bar{Q}p_0^*.
\end{gathered}
\tag{4.3}
$$

Here $\bar{Q} = h_0^+ u_0^+ + h_0^- u_0^-$.

From (4.3), the desired variables η, $h^\pm$, $u^\pm$, $\bar{u}$, $p^* - p_0^* = \Delta p$, and $\bar{q}$ can be expressed as functions of one variable, for example, $Q = \eta\bar{u}$. By virtue of the nonlinearity of the system, these dependences can be ambiguous. For a given value of $Q \geqslant 0$, system (4.3) can be reduced to one equation in terms of the quantity $a = (u^+ + u^-)/2$ as follows.

Let $a \geqslant 0$ be given. Then, $(u^+)^2 - (u^-)^2 = (u_0^+)^2 - (u_0^-)^2$ or $\gamma = \gamma_0 a_0/a$, where $\gamma = (u^+ - u^-)/2$, $\gamma_0 = (u_0^+ - u_0^-)/2$, and $a_0 = (u_0^+ + u_0^-)/2$. Further,

$$
u^+ = a + \gamma, \quad u^- = a - \gamma, \quad h^+ = (h_0^+ u_0^+ - Q/2)/u^+, \quad h^- = (h_0^- u_0^- - Q/2)/u^-,
$$

$$
\eta = H - h^+ - h^-, \quad \bar{u} = Q/\eta, \quad \Delta p = p^* - p_0^* = ((u_0^+)^2 - (u^+)^2)/2.
$$

Substituting the resulting expressions into the conservation law of total momentum, we obtain the equation $P(a, Q) = h^+(u^+)^2 + \eta\bar{u}^2 + h^-(u^-)^2 - h_0^+(u_0^+)^2 - h_0^-(u_0^-)^2 + \Delta pH = 0$ from which the dependence $a = a(Q)$ can be found and the values of the admissible flow parameters ($h^\pm > 0$, $\eta > 0$, and $u^\pm > 0$) are restored. The dependence $q^2 = q^2(Q)$ is determined from the energy equation, and, finally, the distribution of the flow parameters over the x axis can be found from the equation $dQ/dx = 2\sigma q(Q)$.

Velocity Distribution in the Mixing Layer. The distributions of the average quantities in the mixing layer have been found above. In particular, the boundaries $y = h^-(x)$ and $y = h^-(x) + \eta(x)$ of the mixing layer, the velocity of "large vortices" $\bar{q} = \bar{q}(x)$, and the pressure on the upper lid of the channel $p^* = p^*(x)$ are found from Eqs. (4.3). Therefore, for the horizontal and vertical velocity-vector components $u = \tilde{u}(x, y)$ and $v = \tilde{v}(x, y)$, and also for the root-mean-square velocity $q = \tilde{q}(x, y)$, in the boundary-layer approximation the steady-state flow equations (2.7) take the form ($\rho \equiv 1$)

$$
\tilde{u}_x + \tilde{v}_y = 0, \quad \tilde{u}\tilde{u}_x + \tilde{v}\tilde{u}_y - \sigma\tilde{q}\tilde{q}_x + p_x^* = 0, \quad \tilde{u}\tilde{q}_x + \tilde{v}\tilde{q}_y - \sigma\tilde{q}\tilde{u}_y = -\beta\tilde{q}\bar{q}/\eta. \tag{4.4}
$$

Here $\sigma = \sigma_0 \operatorname{sign} \tilde{u}_y$.

It is required to find a solution of (4.4) in the mixing layer $h^-(x) < y < h^-(x) + \eta(x)$, because the large vortices-induced vertical transfer is absent in the potential-flow regions $0 < y < h^-(x)$ and $H - h^+(x) < y < H$ ($\bar{q} = 0$) and the velocity $u = u^\pm(x)$ does not depend on y. The vertical component v in these regions is uniquely restored from the continuity equation and the no-slip condition at the boundaries

$$
\tilde{v}(x, y) = -yu_x^- \quad \text{for} \quad 0 \leqslant y < h^-(x),
$$

$$
\tilde{v}(x, y) = (H - y)u_x^+ \quad \text{for} \quad H - h^+(x) < y \leqslant H,
$$

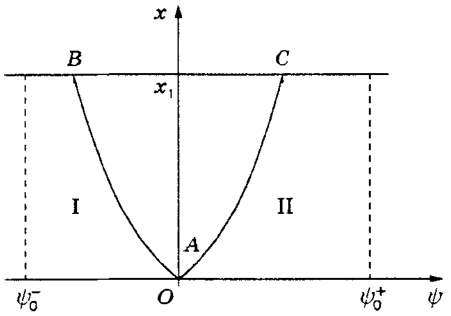

Fig. 3

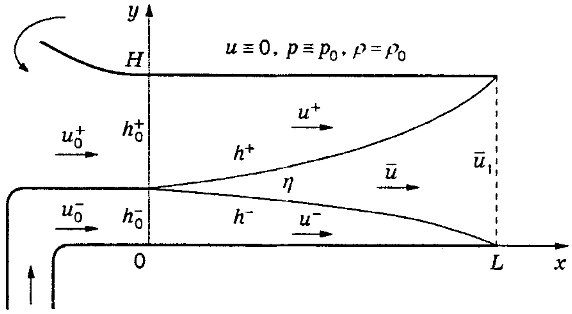

Fig. 4

and turbulence is absent ($\tilde{q} \equiv 0$). Because a continuous solution in the domain $0 < y < H$ is being constructed, the solution of (4.4) is known at the boundary of the mixing region. Let $u_0^+ > u_0^-$ and the velocity profile be monotone ($\tilde{u}_y \geqslant 0$). Then, we have $\sigma \equiv \sigma_0$. For construction of the solution inside the mixing layer, it is convenient to pass to the variables x and ψ (ψ is a stream function). As in the above-considered case of a pressure-gradient-free mixing layer, Eqs. (4.4) become a semilinear system:

$$\tilde{u}_x - \sigma\tilde{q}\tilde{q}_\psi = -p_x^*/\tilde{u}, \qquad \tilde{q}_x - \sigma\tilde{q}\tilde{u}_\psi = -\beta\tilde{q}\tilde{q}/(\eta\tilde{u}). \tag{4.5}$$

A solution of (4.5) is sought in the half-band $x \geqslant 0$, $\psi_0^- \leqslant \psi \leqslant \psi_0^+$, where

$$\psi_0^- = -\int\limits_0^{h_0^-} \tilde{u}(0, y)\, dy = -h_0^- u_0^-, \qquad \psi_0^+ = \int\limits_{h_0^-}^{H} \tilde{u}(0, y)\, dy = h_0^+ u_0^+.$$

The streamline passing through the point A, at which the homogeneous flows merge (Fig. 3), corresponds to the value $\psi = 0$. The boundary of the mixing layer is set by the lines AB and AC. The solution $\tilde{u} = u^-(x)$, $\tilde{q} = \bar{q} = 0$ is known to the left of the line AB (region I). Similarly, the solution has the form $\tilde{u} = u^+(x)$, $\tilde{q} = \bar{q} = 0$ to the right of the line AC (in region II). It is required to find a continuous solution of (4.5) up to the boundary in the region BAC. The lines AB and AC specified by the equations $\psi = \psi^-(x)$ and $\psi = \psi^+(x)$ are the characteristics of system (4.5):

$$\frac{d\psi^-(x)}{dx} = -\sigma\bar{q}(x), \qquad \frac{d\psi^+(x)}{dx} = \sigma\bar{q}(x).$$

Thus, it is required to find a solution of the Goursat problem using the data on the characteristics for the semilinear hyperbolic system (4.5) [$\bar{q}(x) > 0$]. If the function $u^-(x) \geqslant u_{\min} > 0$ in region I ($0 < x < x_1$), the *a priori* estimate $\tilde{u}(x, \psi) \geqslant u_{\min} > 0$ holds in the region BAC for a monotone velocity profile in the mixing layer ($\tilde{u}_\psi \geqslant 0$), and the only difficulty in solving (4.4) is the singularity of the right side of the equations, because $\eta(x) \to 0$ as $x \to 0$. We note that the above solution of system (4.2) has bounded derivatives, i.e., the function p_x^* is bounded as $x \to 0$. Therefore, the pressure-gradient-free, self-similar solution considered in Sec. 3 gives the asymptotic representation of the solution in the neighborhood of the point A.

If a reverse flow [$u^-(x) < 0$] appears as a result of the increase in pressure p^* during the development of a mixing layer in a channel of finite depth, the formulation of the problem changes. In this paper, we omit it.

Problem of an Injector. We consider the problem of a flat injector as an example of the formation of a mixing layer with pressure gradient.

Let an injector be located in a flat channel of depth H and length L (Fig. 4). The channel is immersed into a quiescent incompressible fluid of density $\rho_0 = 1$ and is connected to it freely. A uniform jet of an ideal incompressible fluid of thickness h_0^-, density ρ_0, and velocity u_0^- flows out from the injector in parallel to the bottom of the channel (which can be considered as a plane of symmetry of the flow). As a result of the development of the mixing layer, the steady-state flow is accelerated in the upper layer. It is required

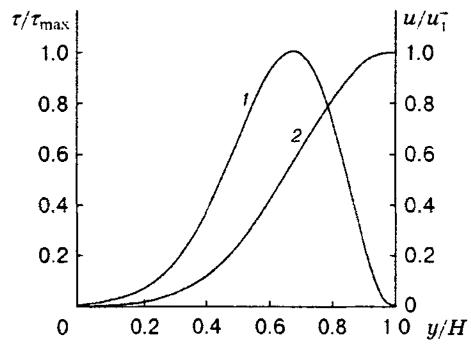

Fig. 5

to determine the flow parameters, the velocity distribution at the exit from the channel, and the "optimal" disposition of the injector, i.e., the values of h_0^-/H and L/H at which the mixing layer covers the entire cross section at the channel exit. To reduce the problem to that considered above, it is sufficient to supplement Eqs. (4.3) by the relation

$$(u_0^+)^2/2 + p_0^* = p_0, \tag{4.6}$$

which follows from the potential-flow condition in the upper layer. Here p_0 is the pressure in the quiescent fluid. The desired quantities are $u^\pm$, $\bar{u}$, $h^\pm$, η, $\Delta p = p^* - p_0^*$, and u_0^+. The conditions $p_1^* = p_0$ and $u^+ = 0$ are satisfied at the channel exit. We do not consider the case where the mixing layer reaches the channel lid ($h_1^+ = 0$), because it becomes an immersed jet in this case. If the channel length L is known, the additional relation

$$L = \int\limits_0^{Q_1} \frac{1}{2\sigma\bar{q}(Q)}\,dQ \tag{4.7}$$

follows from the rate of fluid entrainment into the mixing layer. The dependence of the desired quantities $Q_1 = \eta_1\bar{u}_1$ and $\bar{q} = \bar{q}(Q)$ $(0 < Q < Q_1)$ on the parameter u_0^+ can be found from (4.3) and (4.6), and the dependence $L = L(u_0^+)$ can be found from Eq. (4.7), and then the parameter u_0^+ is defined. Indeed, for $u_1^+ = 0$, we have $(0 < u_0^+ < u_0^-)$

$$Q_1 = \eta_1\bar{u}_1 = 2h_0^+u_0^+, \qquad (u_1^-)^2 = (u_0^-)^2 - (u_0^+)^2,$$

$$h_1^- = \frac{h_0^-u_0^- - Q_1/2}{u_1^-} = \frac{h_0^-u_0^- - h_0^+u_0^+}{u_1^-}, \qquad \Delta p = p_0 - p_0^* = u_0^+/2, \tag{4.8}$$

$$\eta_1 = Q_1/\bar{u}_1, \qquad \bar{u}_1 = \frac{h_0^+(u_0^+)^2 + h_0^-(u_0^-)^2 - h_1^-(u_1^-)^2 - H(u_0^+)^2/2}{2h_0^+u_0^+}.$$

For $0 < Q < Q_1$, the dependence $q^2 = q^2(Q, u_0^+)$ is found from (4.3) similarly to the above-considered mixing layer, and the channel length u_0^+ can be found from (4.7) for a specified value of $L = L(u_0^+)$.

For an "optimal" injector ($h_1^+ = 0$ and $h_1^- = 0$), we derive the following equation for the unknown quantity $z = h_0^-/H$ from (4.8):

$$4z - 1 + z(z - 1/2)/(1 - z)^2 = 0.$$

The single root $z_* \approx 0.267$ can be calculated from this equation. The dimensionless parameters of the flow and the "optimum" length of the channel are then determined from (4.7) and (4.8). In particular, $L/H \simeq 5.38$. Figure 5 shows the mean-velocity (curve 1) and Reynolds-stress (curve 2) distributions at the output cross section of the "optimal" channel for $\beta/(2\sigma) = 6$.

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